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HOMOLOGY, MASSEY PRODUCTS AND MAPS BETWEEN GROUPS

William G. DWYER
Yale University, New Haven, Conn. 06520, U.S.A.

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In a fundamental paper [12], Stallings studied the relationship between the homological behavior of a group homomorphism and its group theoretic qualities. This paper continues his investigation. In Section 1 we obtain a converse to the integral version of his main theorem and in Section 3 a converse to the mod p version (p prime). The cohomological approach of Section 3 also clarifies the relationship between Stallings' work and the Massey product structure of low dimensional group cohomology. Section 2 is transitional: it introduces the notion of higher order Massey product and studies this notion in the integral case. Section 4 applies the ideas of Section 2 in order to sketch the proofs of a few curious theorems, some of them folklore results, and to formulate a technical conjecture.

The difference in viewpoint between Section 1 and Section 3 is explained by the absence of an integral version of (3.3). Unfortunately, the most natural analog of this lemma would imply the notorious dimension subgroup conjecture [10]. However, a rational version of (3.3) is true [4], and in fact the approaches of Section 1 and Section 3 are exactly dual to one another in the rational case. This is left for the reader to work out.

Theorems 1.2 and 3.2 are included to show that the homological filtrations of Section 1 and Section 3 have independent interest. Theorem 1.2 is similar to some results of Stammbach [13]. Also, parts of Section 3 are at least philosophically related to unpublished work of Morgan and Sullivan, in which they investigate the relationship between the fundamental group of a manifold and Massey products in the deRham complex.

If π is a group, $H_*(\pi)$ and $H^*(\pi)$ denote the ordinary group homology and group cohomology of π. Unless otherwise specified, all homology and cohomology is taken with untwisted integer coefficients. The first three sections are purely algebraic, although some topology appears in Section 4.

1. The integral case

If A and B are two subgroups of a group π, $[A,B]$ denotes the subgroup of π generated by all commutators $a^{-1}b^{-1}ab$, $a \in A$, $b \in B$. Recall that the *lower central series* subgroups $\Gamma_k(\pi)$ are the functorial normal subgroups of the group π defined inductively by the formulas:

$$\Gamma_1(\pi) = \pi$$

$$\Gamma_{k+1}(\pi) = [\pi, \Gamma_k(\pi)], \qquad k \geqslant 1 .$$

The group π is said to be *nilpotent* of class $\leqslant k$ if $\Gamma_{k+1}(\pi) = \{1\}$.

For any group π and integer $k \geqslant 2$, $\Phi_k(\pi)$ will denote the kernel of the natural map $H_2(\pi) \to H_2(\pi/\Gamma_{k-1}(\pi))$. The aim of this section is to show that the subgroups $\Phi_k(\pi)$ of $H_2(\pi)$ are closely related to the lower central series quotients of π itself. This relationship is expressed in two main theorems:

Theorem 1.1. *Let $f: \sigma \to \pi$ be a group homomorphism which induces an isomorphism $H_1(\sigma) \to H_1(\pi)$. Then for any $k \geqslant 2$, the following three conditions are equivalent:*
(i) *f induces an epimorphism $H_2(\sigma)/\Phi_k(\sigma) \to H_2(\pi)/\Phi_k(\pi)$.*
(ii) *f induces an isomorphism $\sigma/\Gamma_k(\sigma) \to \pi/\Gamma_k(\pi)$.*
(iii) *f induces an isomorphism $H_2(\sigma)/\Phi_k(\sigma) \to H_2(\pi)/\Phi_k(\pi)$ and an injection $H_2(\sigma)/\Phi_{k+1}(\sigma) \to H_2(\pi)/\Phi_{k+1}(\pi)$.*

Theorem 1.2. *Let A be an abelian group, and let*

$$(*) \qquad 1 \to A \to c \to \pi \to 1$$

be the central extension of π classified by the "k-invariant" α in $H^2(\pi, A)$ [6]. *Let $\bar{\alpha}$ be the image of α under the surjection $H^2(\pi, A) \to \mathrm{Hom}(H_2(\pi), A)$ provided by the universal coefficient theorem. Then for any $k \geqslant 2$*

$$\Gamma_k(\sigma) \cap A = \bar{\alpha}(\Phi_k(\pi)) .$$

It is possible to define the subgroups $\Gamma_k(\pi)$ for transfinite ordinals k as well as for finite ones [12]. The definition of $\Phi_k(\pi)$ then makes sense for arbitrary *successor* ordinals k. Unlike Theorem 1.1, or indeed almost anything else in this paper, (1.2) is valid in this special transfinite case.

The intuitive content of 1.1 and 1.2 is that $\Phi_k(\pi)$ detects all "relations" within π that can be expressed in terms of commutators of length at least k. In fact, it is possible to prove a Hopf theorem which makes this precise: if $F \to \pi$ is a surjection of a free group onto π with kernel R, then $\Phi_k(\pi)$ is naturally isomorphic to

$$[R \cap \Gamma_k(F)]/([F, R] \cap \Gamma_k(F)) .$$

The rest of the section is devoted to proofs.

Proof of 1.1. The proof uses induction on k, and diagram chasing. The statement is easy to see for $k = 2$, in view of the hypothesis on $H_1(f)$. Now suppose that f induces an epimorphism

$$H_2(\sigma)/\Phi_{k+1}(\sigma) \to H_2(\pi)/\Phi_{k+1}(\pi), \qquad k \geqslant 2.$$

Then f gives an epimorphism $H_2(\sigma)/\Phi_k(\sigma) \to H_2(\pi)/\Phi_k(\pi)$ and so, by induction, actual isomorphisms

$$\begin{aligned} H_2(\sigma)/\Phi_k(\sigma) &\to H_2(\pi)/\Phi_k(\pi) \\ \sigma/\Gamma_k(\sigma) &\to \pi/\Gamma_k(\pi)\,. \end{aligned}$$

From this, a five lemma argument shows that the map $\Phi_k(\sigma)/\Phi_{k+1}(\sigma) \to \Phi_k(\pi)/\Phi_{k+1}(\pi)$ is an epimorphism. By the lemma below, there is a diagram of short exact sequences

$$\begin{array}{ccccccccc} 1 \to & \Phi_k(\sigma)/\Phi_{k+1}(\sigma) & \to & \Phi_k(\sigma/\Gamma_k(\sigma)) & \to & \Gamma_k(\sigma)/\Gamma_{k+1}(\sigma) & \to 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \to & \Phi_k(\pi)/\Phi_{k+1}(\pi) & \to & \Phi_k(\pi/\Gamma_k(\pi)) & \to & \Gamma_k(\pi)/\Gamma_{k+1}(\pi) & \to 1\,. \end{array}$$

By the five lemma, the map $\Gamma_k(\sigma)/\Gamma_{k+1}(\sigma) \to \Gamma_k(\pi)/\Gamma_{k+1}(\pi)$ is an isomorphism, and thus, by the nonabelian five lemma, so is the map $\sigma/\Gamma_{k+1}(\sigma) \to \pi/\Gamma_{k+1}(\pi)$. This shows that (i) implies (ii). The other implications are derived in the same way. □

The above proof made use of the curious

Lemma 1.3. *For any group π and integer $k \geqslant 2$, there is a natural short exact sequence*

$$1 \to \Phi_k(\pi)/\Phi_{k+1}(\pi) \to \Phi_k(\pi/\Gamma_k(\pi)) \to \Gamma_k(\pi)/\Gamma_{k+1}(\pi) \to 1\,.$$

Proof. For $k = 2$ the proof is easy. For $k > 2$, the proof consists in chasing an exact diagram:

$$\begin{array}{ccccccccc} & & & 0 & & & & & \\ & & & \downarrow & & & & & \\ & & & \Phi_{k+1}(\pi) & & & & & \\ & & & \downarrow & & & & & \\ 0 \to & \Phi_k(\pi) & \to & H_2(\pi) & \to & H_2(\pi/\Gamma_{k-1}(\pi)) & \to & \Gamma_{k-1}(\pi)/\Gamma_k(\pi) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \Phi_k(\pi/\Gamma_k(\pi)) & \to & H_2(\pi/\Gamma_k(\pi)) & \to & H_2(\pi/\Gamma_{k-1}(\pi)) & \to & \Gamma_{k-1}(\pi)/\Gamma_k(\pi) & \to 0 \\ & & & \downarrow & & & & & \\ & & & \Gamma_k(\pi)/\Gamma_{k+1}(\pi) & & & & & \\ & & & \downarrow & & & & & \\ & & & 0 & & & & & \end{array}$$

Each row or column of this diagram is essentially the low dimensional homology exact sequence of an appropriate group extension [12].

Proof of 1.2. The low dimensional homology exact sequence of the extension (*) is

$$H_2(\sigma) \to H_2(\pi) \to A \to H_1(\sigma) \to H_1(\pi) \to 0 \,.$$

A naturality argument shows that the map $H_2(\pi) \to A$ which appears in this sequence is just $\bar{\alpha}$. This gives a (well-known) proof of 1.2 in the case $k = 2$. The proof in the case $k > 2$ is in three stages.

(a) First we assume that $\bar{\alpha}$ vanishes on $\Phi_k(\pi)$ and show that $\Gamma_k(\sigma) \cap A = \{1\}$. The idea is to exhibit a map of σ into a group of nilpotency class $k-1$ which carries A injectively. Let the abelian group B and the map $\bar{\beta}$ be given as pushouts in the following diagram:

$$\begin{tikzcd} H_2(\pi)/\Phi_k(\pi) \arrow[r, "\bar{\alpha}"] \arrow[d] & A \arrow[d, dashed] \\ H_2(\pi/\Gamma_{k-1}(\pi)) \arrow[r, dashed, "\bar{\beta}"] & B \,. \end{tikzcd}$$

Since the left vertical arrow is an injection, so is the map $A \to B$. Now a short diagram chase using universal coefficient exact sequences and the fact that $H_1(\pi)$ is isomorphic to $H_1(\pi/\Gamma_{k-1}(\pi))$ shows that there is an element β in $H^2(\pi/\Gamma_{k-1}(\pi), B)$ which projects to the given $\bar{\beta}$ in $\mathrm{Hom}(H_2(\pi/\Gamma_{k-1}(\pi)), B)$ and restricts to the member of $H^2(\pi, B)$ which is the image of α under the coefficient homomorphism $A \to B$. Thus there is a map of central extensions

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d] & \sigma \arrow[r] \arrow[d] & \pi \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & B \arrow[r] & \sigma' \arrow[r] & \pi/\Gamma_{k-1}(\pi) \arrow[r] & 1 \end{tikzcd}$$

where σ' is the extension of $\pi/\Gamma_{k-1}(\pi)$ classified by β. It is clear that $\Gamma_k(\sigma') = \{1\}$, since σ' is a central extension of a group of nilpotency class $\leqslant k-2$. Therefore $\Gamma_k(\sigma) \subset \mathrm{kernel}\,(\sigma \to \sigma')$. However, since $A \to B$ is injective, $\mathrm{kernel}\,(\sigma \to \sigma') \cap A = \{1\}$. Thus $A \cap \Gamma_k(\sigma) = \{1\}$.

(b) Next we show that if $\Gamma_k(\sigma) \cap A = \{1\}$, then $\Phi_k(\pi)$ is in the kernel of $\bar{\alpha}$. Consider the map of extensions:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d] & \sigma \arrow[r] \arrow[d] & \pi \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \dfrac{A \cdot \Gamma_{k-1}(\sigma)}{\Gamma_k(\sigma)} \arrow[r] & \dfrac{\sigma}{\Gamma_k(\sigma)} \arrow[r] & \dfrac{\pi}{\Gamma_{k-1}(\pi)} \arrow[r] & 1 \,. \end{tikzcd}$$

The bottom extension is a central extension, since both A and $\Gamma_{k-1}(\sigma)$ are in the center of $\sigma/\Gamma_k(\sigma)$. By naturality there is a commutative square

$$\begin{array}{ccc} H_2(\pi) & \xrightarrow{\bar\alpha} & A \\ \downarrow & & \downarrow \\ H_2(\pi/\Gamma_{k-1}(\pi)) & \longrightarrow & \dfrac{A\cdot\Gamma_{k-1}(\sigma)}{\Gamma_k(\sigma)} \end{array} .$$

Since $A \cap \Gamma_k(\sigma) = \{1\}$, the right vertical arrow is an injection. This means that $\bar\alpha$ must vanish on the kernel of the map $H_2(\pi) \to H_2(\pi/\Gamma_{k-1}(\pi))$, but this kernel is exactly $\Phi_k(\pi)$.

(c) Now consider the case of the general extension (*). There is a diagram

$$\begin{array}{ccccccccc} 1 & \to & A/A\cap\Gamma_k(\sigma) & \to & \sigma' & \to & \pi & \to & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \to & A & \to & \sigma & \to & \pi & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & A/\bar\alpha(\Phi_k(\pi)) & \to & \sigma'' & \to & \pi & \to & 1 \end{array}$$

where σ' and σ'' are the extensions of π classified by the projections of α into $H^2(\pi, A/A \cap \Gamma_k(\sigma))$ and $H^2(\pi, A/\bar\alpha(\Phi_k(\pi)))$. The argument of (a) applies to the bottom extension, and shows that $\Gamma_k(\sigma) \cap A \subseteq \bar\alpha(\Phi_k(\pi))$. The argument of (b) applies to the top extension, and gives the opposite inclusion. This completes the proof.

2. Higher Massey products

This section introduces a group theoretic interpretation of the Massey product structure carried by low dimensional group cohomology and gives a few applications of the interpretation. The results of this section are exploited in Section 3 to give exact analogues of Section 1 in the mod p case.

Some conventions and definitions are necessary. Let R be a ring with unit. The cochain algebra of the group π with coefficients in R, written $C^*(\pi, R)$, is defined to be the R-dual of the reduced bar resolution of $R[\pi]$ (see [6, p. 114]), with cup-product pairing computed by the Alexander–Whitney formula [6, p. 241]. The letter d denotes the coboundary operator in $C^*(\pi, R)$.

Choose classes $\alpha_1, \ldots, \alpha_n$ in $H^1(\pi, R)$, and suppose that there is an array M of cochains

$$M = \{m_{ij} \mid 1 \leqslant i \leqslant n+1, i < j \leqslant n+1, (i,j) \neq (1, n+1), m_{ij} \in C^1(\pi, R)\}$$

such that

(2.1) $\quad m_{i,i+1}$ is a representative of α_i, $1 \leqslant i \leqslant n$.

(2.2) $\quad dm_{ij} = \sum_{k=i+1}^{j-1} m_{ik} \cup m_{kj}, \; j \neq i+1$.

Such an M is called a *defining system* for the nth *order Massey product* $\langle \alpha_1, ..., \alpha_n \rangle$. The *value* of the product relative to this defining system, denoted $\langle \alpha_1, ..., \alpha_n \rangle_M$, is defined to be the element of $H^2(\pi, R)$ represented by the cocycle

$$(2.3) \qquad \sum_{k=2}^{n} m_{1k} \cup m_{k,n+1} \quad \text{(see [8])}.$$

The product $\langle \alpha_1, ..., \alpha_n \rangle$ itself is usually taken to be the subset of $H^2(\pi, R)$ consisting of all elements which can be written in the form $\langle \alpha_1, ..., \alpha_n \rangle_M$ for some defining system M.

Some canonical groups which depend only on the ring R will enter into the interpretation of these products. The group $U(R, n)$ is the multiplicative group of all upper triangular $n \times n$ matrices over R which agree with the identity matrix along the diagonal. The subgroup $Z(R, n)$ of $U(R, n)$ consists of matrices which are identically zero except along the diagonal and at position $(1, n)$. Since $Z(R, n)$ lies in the center of $U(R, n)$, it is possible to define the quotient group $\bar{U}(R, n) = U(R, n)/Z(R, n)$.

Note that a group homomorphism $\phi\colon \pi \to U(R, n)$ is given by a component array ϕ_{ij} $(1 \leqslant i \leqslant n, i < j \leqslant n)$ of set maps $\pi \to R$, which satisfy the identities

$$\phi_{ij}(g_1 g_2) = \phi_{ij}(g_1) + \phi_{ij}(g_2) + \sum_{k=i+1}^{j-1} \phi_{ik}(g_1)\, \phi_{kj}(g_2), \quad g_1, g_2 \in \pi .$$

These identities show that the particular components $\phi_{i,i+1}$, called the *near-diagonal components* of ϕ, are actually group homomorphisms $\pi \to R$, and thus cohomology classes in $H^1(\pi, R)$. A homomorphism $\phi\colon \pi \to \bar{U}(R, n)$ also has near-diagonal components in $H^1(\pi, R)$.

These groups were introduced for the sake of the following theorem:

Theorem 2.4. *Let $\alpha_1, ..., \alpha_n$ be elements of $H^1(\pi, R)$. There is a one-one correspondence $M \leftrightarrow \phi_M$ between defining systems M for $\langle \alpha_1, ..., \alpha_n \rangle$, and group homomorphisms $\phi\colon \pi \to \bar{U}(R, n+1)$ which have $-\alpha_1, ..., -\alpha_n$ as near-diagonal components. Moreover, $\langle \alpha_1, ..., \alpha_n \rangle_M = 0$ in $H^2(\pi, R)$ if and only if the dotted arrow exists in the following diagram*

$$\begin{tikzcd} & U(R, n+1) \arrow[d] \\ \pi \arrow[ur, dashed] \arrow[r, "\phi_M"'] & \bar{U}(R, n+1). \end{tikzcd}$$

Remark. The group $U(R, n+1)$ is a central extension of $\bar{U}(R, n+1)$ by a group isomorphic to the additive group of R. This extension can be pulled back over ϕ_M to give a central extension of π by R. The two-dimensional cohomology class $\langle \alpha_1, ..., \alpha_n \rangle_M$ is exactly the characteristic class of this extension.

The proof of 2.4 appears at the end of this section. With the help of the theorem, it is easy to prove the following result.

Corollary 2.5. *Let $f: \sigma \to \pi$ be a group homomorphism which induces an isomorphism $\sigma/\Gamma_{n+1}(\sigma) \to \pi/\Gamma_{n+1}(\pi)$. Suppose that $\alpha \in H^2(\pi, R)$ can be written as $\langle \alpha_1, \ldots, \alpha_n \rangle_M$ for some $\alpha_i \in H^1(\pi, R)$ and some M. Then $f^*(\alpha) = 0$ iff $\alpha = 0$.*

Proof. This follows immediately from 2.4 and the standard fact that $\Gamma_{n+1}(U(R, n+1))$ is trivial.

For technical reasons (see Section 3) it is sometimes useful to let several coefficient groups act simultaneously in a Massey product. To this end, a *rank n multiplicative system R* (of coefficient groups) is defined to be an array

$$R = \{R_{ij}: 1 \leqslant i \leqslant n+1, i < j \leqslant n+1\}$$

of abelian groups, together with pairings

$$\mu: R_{ij} \otimes R_{jk} \to R_{ik}$$

which are associative in the sense that, for all meaningful choices of indices, the two induced maps

$$R_{ij} \otimes R_{jk} \otimes R_{kl} \to R_{il}$$

agree. Given any such array, there are induced cup product pairings $C^*(\pi, R_{ij}) \otimes C^*(\pi, R_{jk}) \to C^*(\pi, R_{ik})$. Here, as before, $C^*(\pi, -)$ denotes the appropriate dual of the reduced bar resolution, and cup product pairings are to be computed using the Alexander–Whitney formula

Given a rank n multiplicative system R, choose classes $\alpha_i \in H^1(\pi, R_{i,i+1})$, $1 \leqslant i \leqslant n$, and suppose that there is an array M of cochains

$$M = \{m_{ij} \mid 1 \leqslant i \leqslant n+1, i < j \leqslant n+1, (i,j) \neq (1, n+1), m_{ij} \in C^1(\pi, R_{ij})\}$$

which satisfy (2.1) and (2.2). Such an M is called a *defining system* for the (generalized) nth order Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle^R$. As before, the *value* of this product, relative to M, denoted $\langle \alpha_1, \ldots, \alpha_n \rangle^R_M$, is taken to be the cohomology class in $H^2(\pi, R_{1,n+1})$ of the cocycle given by (2.3).

Now let $U(R)$ be the group of arrays

$$\{r_{ij} \in R_{ij}, 1 \leqslant i \leqslant n+1, i < j \leqslant n+1\},$$

with group operation given by the formula

$$\{r_{ij}\} \cdot \{r'_{ij}\} = \{s_{ij}\}$$

$$s_{ij} = r_{ij} + r'_{ij} + \sum_{k=i+2}^{j-1} \mu(r_{ik} \otimes r'_{kj}).$$

The groups $Z(R)$ and $\bar{U}(R) = U(R)/Z(R)$ are defined in the obvious way. As before, the *near-diagonal components* of a group map $\pi \to U(R)$ or $\pi \to \bar{U}(R)$ are taken to be the evident cohomology classes. The following theorem is immediate (in view of the proof of 2.4):

Theorem 2.6. *Suppose $\alpha_i \in H^1(\pi, R_{i,i+1})$, $1 \leqslant i \leqslant n$. There is a one-one correspondence $M \leftrightarrow \phi_M$ between defining systems M for $\langle \alpha_1, ..., \alpha_n \rangle^R$ and group homomorphisms $\phi : \pi \to \bar{U}(R)$ which have $-\alpha_1, ..., -\alpha_n$ as near-diagonal components. Moreover, $\langle \alpha_1, ..., \alpha_n \rangle_M^R = 0$ in $H^2(\pi, R_{1,n+1})$ iff the dotted arrow exists in the following diagram*

$$\begin{tikzcd} & U(R) \arrow[d] \\ \pi \arrow[ur, dashed] \arrow[r, "\phi_M"'] & \bar{U}(R) . \end{tikzcd}$$

Suppose that A is an abelian group. A class α in $H^2(\pi, A)$ is said to be *decomposable in terms of* nth *order Massey products* if there is some rank n multiplicative system R with $R_{1,n+1} = A$, some choice of classes $\alpha_i \in H^1(\pi, R_{i,i+1})$ and some defining system M for $\langle \alpha_1, ..., \alpha_n \rangle^R$ such that $\alpha = \langle \alpha_1, ..., \alpha_n \rangle_M^R$. The set $\Phi^n(\pi, A)$ is defined to be the subset of $H^2(\pi, A)$ consisting of all such decomposable classes. By constructing suitable R and M it is not hard to show that $\Phi^n(\pi, A)$ is actually a subgroup of $H^2(\pi, A)$. The following corollary is a generalization of 2.5 and a partial dual to 1.1.

Corollary 2.7. *Let $f: \sigma \to \pi$ be a group homomorphism which induces an isomorphism $\sigma/\Gamma_{n+1}(\sigma) \to \pi/\Gamma_{n+1}(\pi)$. Then for every abelian group A, f induces an isomorphism $\Phi^n(\pi, A) \to \Phi^n(\sigma, A)$ and an epimorphism $\Phi^{n+1}(\pi, A) \to \Phi^{n+1}(\sigma, A)$.*

Proof. It is only necessary to note that if R is a rank k multiplicative system, then $\Gamma_k(\bar{U}(R)) = \Gamma_{k+1}(U(R)) = \{1\}$.

The following corollary is an extension of one part of the cup product reduction theorem. It again shows that the cohomology filtration introduced here is dual or perpendicular to the homology filtration of Section 1.

Corollary 2.8. *Let $\alpha(x)$ denote the Kronecker product of the class α in $H^2(\pi, A)$ and the homology class x in $H_2(\pi)$. Then $\alpha(x) = 0$ if $\alpha \in \Phi^n(\pi, A)$ and $x \in \Phi_{n+1}(\pi)$.*

Proof. Let $f: \pi \to \pi/\Gamma_n(\pi)$. By 2.7 for any α in $\Phi^n(\pi, A)$, there is a class $\alpha' \in \Phi^n(\pi/\Gamma_n \pi, A)$ such that $f^*(\alpha') = \alpha$. Pick $x \in \Phi_{n+1}(\pi)$. Then

$$\alpha(x) = [f^*(\alpha')]\,(x) = \alpha'(f_*(x)) = \alpha'(0) = 0 .$$

It only remains to give the

Proof of 2.4. If x is an i-dimensional chain of the reduced bar resolution of $Z[\pi]$, $c(x)$ will denote the Kronecker product of x with the cochain $c \in C^i(\pi, R)$. Suppose that $M = \{m_{ij}\}$ is a defining system for $\langle\alpha_1, ..., \alpha_n\rangle$. Take the Kronecker product of both sides of (2.2) with a basic two-chain $[g_1 | g_2]$ $(g_1, g_2 \in \pi)$ of the reduced bar resolution:

$$dm_{ij}([g_1|g_2]) = \sum_{k=i+1}^{j-1} (m_{ik} \cup m_{kj})([g_1|g_2]) .$$

Apply the Alexander–Whitney formula, and the formula for the boundary in the bar resolution, to get

$$m_{ij}([g_1]) + m_{ij}([g_2]) - m_{ij}([g_1 g_2]) = \sum_{k=i+1}^{j-1} m_{ik}([g_1])\, m_{kj}([g_2])$$

where $[g]$ $(g \in \pi)$ denotes a basic one-chain of the bar resolution. Now let $\phi_{ij} = -m_{ij}$ and write g for $[g]$. The formula becomes

$$\phi_{ij}(g_1 g_2) = \phi_{ij}(g_1) + \phi_{ij}(g_2) + \sum_{k=i+1}^{j-1} \phi_{ik}(g_1)\, \phi_{kj}(g_2) .$$

These ϕ_{ij} are evidently the components of a homomorphism $\phi_M : \pi \to \bar{U}(R, n+1)$. Moreover, given any homomorphism $\phi : \pi \to \bar{U}(R, n+1)$ with $-\alpha_1, ..., -\alpha_n$ as near-diagonal components, it is easy to reverse the above process and construct a defining system for $\langle\alpha_1, ..., \alpha_n\rangle$. The proof that $\langle\alpha_1, ..., \alpha_n\rangle_M = 0$ iff ϕ_M can be lifted to $U(R, n+1)$ is a standard cochain calculation.

3. The mod p case

Let p be a prime number, and let F be the finite field of integers modulo p. The purpose of this section is to prove "mod F" versions of 1.1 and 1.2. These modular theorems will be stated and proved from a cohomological point of view, partially because Section 2 provides an easy way to interpret the Massey products that come up in this setting.

First of all, we need mod F versions of the various filtrations studied in Section 1. The correct group theoretic notion is the filtration of a group π by its *restricted mod p lower central series* subgroups $\Gamma_k^F(\pi)$. By definition,

$$\Gamma_k^F(\pi) = \langle [g_1, ..., g_n]^{p^m} \mid p^m n \geqslant k, g_i \in \pi \rangle$$

where $[., ..., .]$ denotes simple commutator and the brackets mean the smallest subgroup of π containing the given elements. Note that $\Gamma_k^F(\pi)$ is *not* the most rapidly descending central series for π whose successive quotients are F-modules.

The proper cohomological filtration will be defined in the language of Section 2. Let A be a vector space over F. An element α in $H^2(\pi, A)$ is said to be decomposable over F in terms of kth order Massey products if there is some rank k multiplicative system R of F-modules with $R_{1,k+1} = A$, some choice of classes $\alpha_i \in H^1(\pi, R_{i,i+1})$, and some defining system M for $\langle\alpha_1, ..., \alpha_k\rangle^R$, such that $\alpha = \langle\alpha_1, ..., \alpha_k\rangle^R_M$. The set $\Phi^k_F(\pi, A)$ is defined to be the subset of $H^2(\pi, A)$ made up of all such F-decomposable classes. By convention, $\Phi^k_F(\pi) = \Phi^k_F(\pi, F)$. As before, it is not hard to see that $\Phi^k_F(\pi, A)$ is actually a sub F-module of $H^2(\pi, A)$, and that $\Phi^k_F(\pi, A) \subseteq \Phi^{k+1}_F(\pi, A)$.

The main theorems of this section are

Theorem 3.1. *Let $f : \sigma \to \pi$ be a group homomorphism which induces an isomorphism $H^1(\pi, F) \to H^1(\sigma, F)$. Then, for $k \geqslant 2$, the following three conditions are equivalent:*
(i) *f induces an injection $\Phi^{k-1}_F(\pi) \to \Phi^{k-1}_F(\sigma)$.*
(ii) *f induces an isomorphism $\sigma/\Gamma^F_k(\sigma) \to \pi/\Gamma^F_k(\pi)$.*
(iii) *f induces an isomorphism $\Phi^{k-1}_F(\pi) \to \Phi^{k-1}_F(\sigma)$ and an epimorphism $\Phi^k_F(\pi) \to \Phi^k_F(\sigma)$.*

Theorem 3.2. *Let A be an F-module, and suppose that*

$$(*) \qquad 1 \to A \to \sigma \to \pi \to 1$$

is the central extension of π classified by the element α in $H^2(\pi, A)$. Suppose that $s : A \to F$ is a homomorphism, and let $s_*(\alpha)$ denote the image of α in $H^2(\pi, F)$ under the coefficient map given by s. Then
(i) $(\Gamma^F_k(\sigma) \cap A) \subseteq \text{kernel}(s) \leftrightarrow s_*(\alpha) \in \Phi^{k-1}_F(\pi)$ *or, alternatively*
(ii) $\Gamma^F_k(\sigma) \cap A = \bigcap \text{kernel}(s)$
where the intersection is taken over all $s : A \to F$ such that $s_*(\alpha) \in \Phi^{k-1}_F(\pi)$.

It is possible to prove 3.2 more or less directly from 3.1.

Proof of 3.2. First of all, suppose that A is a one-dimensional F-module, and pick $s: A \to F$. Assume without loss that s is not the zero map; thus s must be injective, and (i) becomes

$$\Gamma^F_k(\sigma) \cap A = \{1\} \leftrightarrow s_*(\alpha) \in \Phi^{k-1}_F(\pi)\,.$$

The left hand condition is equivalent to the statement that the projection map $\sigma/\Gamma^F_k(\sigma) \to \pi/\Gamma^F_k(\pi)$ is *not* an isomorphism. According to 3.1, this map *is* an isomorphism iff
(a) $H^1(\pi, F) \to H^1(\sigma, F)$ is an isomorphism and
(b) $\Phi^{k-1}_F(\pi) \to \Phi^{k-1}_F(\sigma)$ is an injection.
Thus we are reduced to showing that one of these conditions fails iff $s(\alpha) \in \Phi^{k-1}_F(\pi)$. The low dimensional cohomology exact sequence of (*) is

$$0 \to H^1(\pi, F) \to H^1(\sigma, F) \to \text{Hom}(A, F) \to H^2(\pi, F) \to H^2(\sigma, F)\,.$$

The usual argument shows that the map $\text{Hom}(A, F) \to H^2(\pi, F)$ takes the homomorphism s to the class $s_*(\alpha)$. Suppose that $s_*(\alpha) \in \Phi^{k-1}_F(\pi)$. Then, since $s \neq 0$, it is clear from the exact sequence that either (a) or (b) is violated. Conversely, if

(a) or (b) fails, then $s_*(\alpha) \in \Phi_F^{k-1}(\pi)$ for some non-zero $s: A \to F$, and therefore, by the one-dimensionality of $\mathrm{Hom}(A, F)$, for all.

Now consider a general A and non-zero $s: A \to F$. Construct the extension σ' of π by F classified by $s_*(\alpha)$, and note that there is a map of extensions

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d, "s"] & \sigma \arrow[r] \arrow[d] & \pi \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & F \arrow[r] & \sigma' \arrow[r] & \pi \arrow[r] & 1 . \end{tikzcd}$$

By the above argument, $s_*(\alpha) \in \Phi_F^{k-1}(\pi)$ iff $F \cap \Gamma_k^F(\sigma') = \{1\}$. This last is true, by a diagram chase, iff $\mathrm{kernel}(s)/(\mathrm{kernel}(s) \cap \Gamma_k^F(\sigma))$ does not surject onto $A/(A \cap \Gamma_k^F(\sigma))$. Since $\mathrm{kernel}(s)$ is of codimension 1 in the F-module A, this happens iff $\mathrm{kernel}(s) \supseteq A \cap \Gamma_k^F(\sigma)$.

There only remains to give the

Proof of 3.1. The proof that (ii) implies (i) and (iii) is the same as the proof of 2.5 or 2.7. It is only necessary to see that if R is a rank n multiplicative system of F-modules, then $\Gamma_n^F(\bar{U}(R)) = \Gamma_{n+1}^F(U(R)) = \{1\}$.

Now we will show that (i) implies (ii). An elementary argument, using the fact that f gives an isomorphism $H^1(\pi, F) \to H^1(\sigma, F)$, shows that the map $\sigma/\Gamma_k^F(\sigma) \to \pi/\Gamma_k^F(\pi)$ induced by f must be an epimorphism. By inducting on k, we can assume that $\sigma/\Gamma_{k-1}^F(\sigma) \to \pi/\Gamma_{k-1}^F(\pi)$ is an isomorphism. What is left is to show that $\sigma/\Gamma_k^F(\sigma) \to \pi/\Gamma_k^F(\pi)$ is injective. The key to this is

Lemma 3.3. *There is a rank $k-1$ multiplicative system R of F-modules and an injective group homomorphism $\phi: \sigma/\Gamma_k^F(\sigma) \to U(R)$.*

Assume 3.3 for the moment. From the inductive hypothesis it is clear that there is a commutative diagram of solid arrows

$$\begin{tikzcd} \sigma/\Gamma_k^F(\sigma) \arrow[r, "\phi"] \arrow[d] & U(R) \arrow[d] \\ \pi/\Gamma_k^F(\pi) \arrow[r] \arrow[ur, dashed] & \bar{U}(R) \end{tikzcd}$$

where R and ϕ are as in 3.3. By 2.6, the obstruction to finding a dotted arrow which would make the lower triangle commute is a class α in $\Phi_{k-1}^F(\pi, A)$, where $A = R_{1,k}$. Clearly $f^*(\alpha) = 0$ in $\Phi_{k-1}^F(\sigma, A)$. By 3.4 below, α itself is zero. A short argument involving the fact that f induces an isomorphism $H^1(\pi, A) \to H^1(\sigma, A)$ now shows that a dotted arrow can actually be chosen to make both triangles of the diagram commute. Thus since ϕ is injective, the map $\sigma/\Gamma_k^F(\sigma) \to \pi/\Gamma_k^F(\pi)$ is also injective.

The proof above used the following

Lemma 3.4. *Let f be as in 3.1, and suppose that f induces an injection $\Phi_F^{k-1}(\pi) \to \Phi_F^{k-1}(\sigma)$. Then f induces an injection $\Phi_F^{k-1}(\pi, A) \to \Phi_F^{k-1}(\sigma, A)$ for any F-module A.*

Proof. This follows from the fact that $\Phi_F^{k-1}(\pi, A)$ is a functor of π and A, together with the fact that any F-module A is a direct sum of a number of copies of F itself.

Proof of 3.3. Let $F[\sigma]$ be the group ring of σ over F, and $I \subseteq F[\sigma]$ the augmentation ideal. For $k \geqslant 1$, I^k is the kth power of I; by convention, $I^0 = F[\sigma]$. Let $\mathcal{F}$ be the category of vector spaces over F. Construct a rank $k-1$ multiplicative system R of F-modules by setting

$$R_{ij} = \mathrm{Hom}_{\mathcal{F}}\,(I^{i-1}/I^i, I^{j-1}/I^j)$$

and letting the structure maps be induced by composition of homomorphisms. It is not hard to identify $U(R)$ with the group of all $\mathcal{F}$-maps $\Psi: F[\sigma]/I^k \to F[\sigma]/I^k$ which have the property that $\Psi(x) - x \in I^{j+1}/I^k$ if $x \in I^j/I^k$. Under this identification, the left translation action of σ on $F[\sigma]$ gives a homomorphism $\sigma \to U(R)$. The fact that this induces a faithful homomorphism $\phi: \sigma/\Gamma_k^F(\sigma) \to U(R)$ follows at once from Jenning's theorem [3, 11] that $g \in \Gamma_k^F(\sigma)$ iff $(1-g) \in I^k$.

4. Applications

Let σ be a free group on n generators $x_1, \ldots, x_n$ and let P be the power series algebra over $\mathbf{Z}$ on n non-commuting indeterminates $k_1, \ldots, k_n$. Magnus [7] has studied the homomorphism Ψ from σ into the multiplicative group of units of P defined by setting $\Psi(x_i) = 1 + k_i$ $(1 \leqslant i \leqslant n)$. Pick $g \in \sigma$, and write

$$\Psi(g) = 1 + \sum \mu(i_1, \ldots, i_j; g)\, k_{i_1} \cdots k_{i_j}\,.$$

The integers $\mu(i_1, \ldots, i_j; g)$ are called the *Magnus coefficients* of g; the number j is called the *degree* of the Magnus coefficient. A well-known theorem [7] says that $g \in \Gamma_k(\sigma)$ iff all the Magnus coefficients of g with degree less than k vanish.

Let $f: \sigma \to \pi$ be a surjection with kernel ρ, so that f is a free presentation of π A classical theorem of Hopf gives a surjection $\rho \cap \Gamma_2(\sigma) \to H_2(\pi)$, written $r \mapsto \{r\}$.

The following proposition points the way toward Stallings' conjectured homological interpretation of Milnor's link invariants [9]. We use the convention that the image of an element x of π or σ in $H_1(\pi)$ is denoted by the same letter. Kronecker products are written as in Section 2.

Proposition 4.1. *Suppose that $r \in \rho \cap \Gamma_k(\sigma)$ for $k \geqslant 2$. Let $\alpha_1, \ldots, \alpha_k$ be elements of $H^1(\pi)$ and suppose that M is a defining system for $\langle \alpha_1, \ldots, \alpha_k \rangle$. Then*

$$\langle \alpha_1, \ldots, \alpha_k \rangle_M(\{r\}) = \sum (-1)^{k+1} \alpha_1(x_{i_1}) \cdots \alpha_k(x_{i_k})\, \mu(i_1, \ldots, i_k; r)$$

where the sum is over all k-tuples $(i_1, ..., i_k)$ *of integers* $\leqslant n$.

The proof of this, which we do not give in detail, follows directly from the proof of 2.4, the definition of the Hopf map, and the following algebraic lemma

Lemma 4.2. *Let* σ *be as above, and suppose that* $\phi: \sigma \to U(\mathbf{Z}, k+1)$ *is a homomorphism with components* ϕ_{ij} $(1 \leqslant i \leqslant k+1, i < j \leqslant k+1)$. *Let* $\alpha_i = \phi_{i,i+1}$ $(1 \leqslant i \leqslant k)$ *and pick* $r \in \Gamma_k(\sigma)$. *Then*

$$\phi_{ij}(r) = 0, (i,j) \neq (1, k+1)$$

$$\phi_{1,k+1}(r) = \sum \alpha_1(x_{i_1}) \cdots \alpha_k(x_{i_k})\, \mu(i_1, ..., i_k; r)$$

where the sum is as in 4.1.

The Massey product $\langle \alpha_1, ..., \alpha_k \rangle$ is said to *contain zero* if it admits a defining system M such that $\langle \alpha_1, ..., \alpha_k \rangle_M = 0$. The following proposition is another direct consequence of 4.2.

Proposition 4.3. *Let* $\alpha_1, ..., \alpha_n$ *be n elements of* $H^1(\pi)$ *such that all* kth *order products of the form* $\langle \alpha_{i_1}, ..., \alpha_{i_k} \rangle$ $(1 \leqslant i_j \leqslant n)$ *contain zero. Let* σ *be as above, and suppose that there is a map* $f: \sigma \to \pi$ *such that* $(f^*(\alpha_i))(x_j) = \delta_{ij}$ *(Kronecker delta)* $1 \leqslant i, j \leqslant n$. *Then* $\mathrm{kernel}(f) \subseteq \Gamma_{k+1}(\sigma)$.

Now suppose that $\alpha_1, ..., \alpha_n$ are classes in $H^1(\pi)$ with the property that *all* Massey products between them contain zero. Suppose further that these classes are linearly independent, so that there are elements $y_1, ..., y_n$ in π with $\alpha_i(y_j) = \delta_{ij}$, $1 \leqslant i, j \leqslant n$. In view of the fact that free groups are residually nilpotent, 4.3 implies that $y_1, ..., y_n$ must generate a rank n free subgroup of π. It is curious to note that the following similar result can be proved directly in a geometric way.

Proposition 4.4. *Let K be a semi-simplicial complex, and suppose that* $a, b \in C^1(K, \mathbf{Z})$ *are two cocycles which represent linearly independent cohomology classes. Then, if the cup product of a and b vanishes on the cochain level,* $\pi_1(K)$ *maps onto a free group on two generators.*

Proof. It is enough to construct a map from the geometric realization of K into the one-point union of two circles which induces an epimorphism on H_1. The map is defined to take each zero simplex of the geometric realization to the basepoint of $S^1 \vee S^1$, and each one-simplex x to a path which winds around the first S^1 $b(x)$ times and the second S^1 $a(x)$ times. The details are routine.

Another corollary of 4.2 or 4.3 is a criterion for a group to be parafree in the sense of Baumslag. For our purpose a group π is *parafree* [1,13] if there is a free group σ and a map $f: \sigma \to \pi$ which induces isomorphisms $\sigma/\Gamma_k(\sigma) \to \pi/\Gamma_k(\pi)$ for all $k \geq 2$.

Corollary 4.5. *A group π is parafree if and only if $H_1(\pi)$ is free abelian and every Massey product of classes in $H^1(\pi)$ contains zero.*

Finally, the known identities satisfied by Magnus coefficients [2] together with 4.1 suggest the following conjecture about general Massey products. For the statement, let S be the set of positive integers $\leq p+q$, and recall that a (p, q) shuffle is a bijection $f: S \to S$ such that $f(i) < f(j)$ if $i < j$ and $j \leq p$ or $i \geq p + 1$.

Conjecture 4.6. *Suppose that X is a space, and that $\alpha_1, \ldots, \alpha_{p+q}$ are odd dimensional classes in $H^*(X)$. Then, under the usual hypotheses* [8]

$$0 \in \sum \langle \alpha_{f^{-1}(1)}, \ldots, \alpha_{f^{-1}(p+q)} \rangle$$

where the sum is taken over all (p, q) shuffles f.

The conjecture has been stated only for odd dimensional classes to avoid introducing complicated signs. These shuffle identities seem to be more general than the identities studied by May [8] or Kraines [5], but they also seem more complicated to prove.

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